Homework due Monday, September 29

Definition and Notation: Let Γ be any set of formulae. Recall that the notation $\Gamma \models \phi$ means that, for every structure $\mathcal{M}$ such that $\mathcal{M} \models \Gamma$, it also holds that $\mathcal{M} \models \phi$. The notation $\Gamma \vdash \phi$ means that there is a proof (using some standard notion of "proof system") such that every line of the proof is either an element of Γ or else follows from some earlier lines according to the inference rules of the proof system, where the final line in the proof is ϕ.

Gödel's Completeness Theorem says that $\Gamma \models \phi$ implies $\Gamma \vdash \phi$.

2. Using Gödel's Completeness Theorem, prove that if, for every finite subset $\Gamma' \subset \Gamma$ there is a structure $\mathcal{M}_{\Gamma'}$ such that $M_{\Gamma'} \models \Gamma'$, then there is a structure $\mathcal{M}$ such that $\mathcal{M} \models \Gamma$. (Hint: If there is not a structure $\mathcal{M}$ such that $\mathcal{M} \models \Gamma$, then $\Gamma \models (\phi \wedge \neg\phi)$ for any formula ϕ.)

3. Consider some set of axioms Γ_1 such that $\mathbb{N} \models \Gamma_1$. Let **c** be a new constant. Consider $\Gamma_2 = \Gamma_1 \cup \{$**c**$> 1$, **c**$> 1 + 1$, **c**$> 1 + 1 + 1$,...$\}$. Prove that there is a structure $\mathcal{M}$ such that $\mathcal{M} \models \Gamma_2$.

$\mathcal{M}$ is called a *non-standard model of arithmetic*. Note that $\mathcal{M}$ contains elements that are larger than any "standard" integer.

4. Let Γ be some set of axioms that are satisfied by the Natural Numbers $\mathbb{N}$ (where, in particular, Γ should require that, for every x and y such that $x \neq y$, either $x < y$ or $y < x$, and Γ should be "reasonable" in the sense that there is a program that can tell if something is an axiom or not). Let $\phi(x, y, c, t)$ be a formula (in the signature of the first-order theory of $\mathbb{N}$) such that, for every tuple of closed terms (**x,y,c,t**), it holds that $\mathbb{N} \models \phi$(**x,y,c,t**) if and only if **c** is a number whose binary representation encodes a sequence of $\mathbf{t}$ configurations $C_1, C_2, \ldots, C_{\mathbf{t}}$ such that C_1 encodes the initial configuration of Turing machine $M_{\mathbf{x}}$ on input $\mathbf{y}$, and $C_{\mathbf{t}}$ is a halting configuration. (The existence of such a formula ϕ follows from some proofs of Gödel's *incompleteness* theorem, which shows that there is a Turing machine $M_{\mathbf{x}}$ and an input **y** such that $M_{\mathbf{x}}$ does not halt on input **y**, but there is no proof from Γ of the (true) statement $\neg\exists t \exists c \phi$(**x,y**$, c, t$) – because otherwise there would be an algorithm to solve the halting problem.) Let ψ denote the statement $\exists t \exists c \phi$(**x,y**$, c, t$).

Show that there must be a non-standard model $\mathcal{M}$ (that also satisfies Γ) such that $\mathcal{M} \models \psi$.